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ESTIMATING THE ERROR OF LOCAL INTERPOLATION SPLINE FUNCTIONS IN THE CLASS OF $W^1[a_1, b_1]$ FUNCTIONS AND CREATING A DIGITAL PROCESSING MODEL OF GEOPHYSICAL SIGNALS.

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Abstract: In the modern scientific landscape, mathematical modeling plays a vital role in addressing challenges across various industrial domains. Among these, spline functions have emerged as effective tools, offering several advantages over classical interpolation methods [1,3,9]. This study focuses on error analysis and practical applications of a local interpolation cubic spline, constructed using a linear combination of parabolas with two shared points. Using experimentally acquired geophysical signal data, the error behavior of the spline model is examined through numerical analysis and graphical validation. The results confirm the model's effectiveness in accurate signal reconstruction.

Keywords: Spline function, signal processing, geophysical signal

1. Introduction:

Local interpolation cubic splines are widely used in digital processing of geophysical, biomedical, and other types of signals. They also support approximate evaluations of regular and singular integrals, as well as integral equations. Prior works [1–5] have shown that such splines are well-suited for developing efficient quadrature formulas and digital models. Notable types include Hermite splines, which interpolate both function values and derivatives, and Ryabenkov splines [1,4,7,9], which interpolate function values at grid nodes. While Ryabenkov cubic splines typically achieve an approximation order of $O(h^2)$, the local cubic spline model discussed in this paper reaches a higher accuracy of $O(h^3)$, beyond which the interpolation accuracy saturates. Based on the construction procedure of the spline function with defect equal to 2 as given in [Scopus article link], we can determine the following from the conditions for determining α_1 , α_2 , α_3 and α_4 : [2,8,11]:

$$\alpha_1 + \alpha_3 = 1, \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \quad (1)$$

determining that $\alpha_3 = 1 - \alpha_1$, $\alpha_4 = -\alpha_2$. from above [3,5,7].

After the condition of joining the first and second derivatives of splines $S'_i(x)$, $S'_{i+1}(x)$ and $S''_i(x)$, $S''_{i+1}(x)$ at the node, after simplifications we have the following:

$$\alpha_1(\Delta^2 f_{i+1} - \Delta^2 f_{i-1}) + \alpha_2 \Delta^3 f_{i-1} = \Delta^3 f_i, \quad (2)$$

$$\alpha_1(\Delta^4 f_{i-1}) - \alpha_2(\Delta^2 f_{i-1} + \Delta f_i - \Delta f_{i+2}) = \Delta^3 f_i. \quad (3)$$

where Δ is the finite difference operator.

If we take the solutions of the system of equations (4)-(6) as $\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*$ then

$$S_3(x) = S_3(t) = (\alpha_1^* + \alpha_2^*t)y_i(t) + (\alpha_3^* + \alpha_4^*t)y_{i+1}(t)$$

the defect is a spline equal to 1. But there are complex rational functions depending on the coefficients $f_{i-1}, f_i, f_{i+1}, f_{i+2}, f_{i+3}$. Therefore, such a spline is inconvenient for practical use. If we take $\alpha_1 = \frac{5}{6}, \alpha_2 = -1$, we get $\alpha_3 = \frac{1}{6}, \alpha_4 = 1$. These values satisfy conditions (1)-, (2)-, but do not satisfy equation (3). It follows that we have an interpolation spline with 2 defects in the section $[x_i, x_{i+1}]$.

$$S_3(f; x) = S_3(t) = (1-t)y_i(t) + ty_{i+1}(t). \quad (4)$$

here:

$$\begin{aligned} \varphi_1(t) &= \frac{11t^2}{12} - \frac{5t}{12} - \frac{t^3}{2} & \varphi_2(t) &= 1 - \frac{9t^2}{4} - \frac{t}{4} + \frac{3t^3}{2} \\ \varphi_3(t) &= \frac{7t^2}{4} + \frac{3t}{4} - \frac{3t^3}{2} & \varphi_4(t) &= \frac{t^3}{2} - \frac{5t^2}{12} - \frac{t}{12} \end{aligned}$$

$$t = (x - x_i)/h, \quad h = x_{i+1} - x_i, \quad x \in [x_i, x_{i+1}], \quad t \in [0, 1]$$

[silka] maqollarda keltirilgan parabolik $y_i(t)$ va $y_{i+1}(t)$ tenglamalar ifodasini (4) tenglikka qo'yib, $[x_i, x_{i+1}]$ kesmada quyidagiga ega bo'lamiz:

$$S_3(f; x) = \sum_{j=0}^3 \varphi_{j+1}(t) f(x_{i+j-1}). \quad (5)$$

2. Some known results and auxiliary definitions

Let the mesh nodes be given in the section $[a, b]: \Delta: a = x_0 < x_1 < \dots < x_N = b$. Here, P_m is a set of polynomials of degree no greater than m , and $C^k = C^k[a, b]$ is a class of functions with ordered continuous derivatives. $C^{-1}[a, b]$ is a class of continuous functions without discontinuity of the first type. The function $S_m(x) = S_{m,k}(x, \Delta)$ is called a polynomial spline of degree k ($0 \leq k \leq m-1$) with defective $\{x_i\}$ nodes, if the following conditions hold [1,3,5,11]:

- $S_m(x) \in P_m$ на $[x_i, x_{i+1}]$, $i = 0, 1, \dots, N-1$;
- $S_m(x) \in C^{m-k}[a, b]$.

Suppose that the values of the function $f(x)$ $f_i = f(x_i)$ at nodes x_i of the DDD network are known. A spline $S_m(x)$ is said to be interpolated if the following conditions are met:

- $S_m(x_i) = f_i$, $i = 0, 1, \dots, N$.

A measurable function $f(x)$ is said to be bounded on the section $[a, b]$ if there exists μ such that the dimension of the set of points for which the condition $|f(x)| > \mu$ is fulfilled is zero [7,11]. The smallest of those satisfying this condition is defined as $\text{ess sup}_{x \in [a,b]} |f(x)|$. $L_\infty[a, b]$ is a measurable

and bounded space, and the norm in it is defined as follows:

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$$\|f(x)\|_{L_\infty[a,b]} = \text{ess sup}_{x \in [a,b]} |f(x)|.$$

The class of functions $f(x)$ with an absolute continuous 1-1 and 1-order derivative in $L_\infty[a, b]$ on the interval $[a, b]$ is defined as $W^1[a, b]$ [6].

Next, we apply the generalized version of the mean value theorem below. Let $a \leq x_1 \leq x_2 \leq \dots \leq x_k \leq b$ and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ values have the same sign, and $f(x) \in C[a, b]$.

So

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_k f(x_k) = (\lambda_1 + \lambda_2 + \dots + \lambda_k) f(\xi), \quad a \leq \xi \leq b. \quad (6)$$

We also use the Taylor series distribution:

$$f(x) = f(a) + \dots + \frac{f^{(r-1)}(a)(x-a)^{r-1}}{(r-1)!} + \frac{1}{(r-1)!} \int_a^x (x-v)^{r-1} f^{(r)}(v) dv, \quad (7)$$

or

$$f(x) = f(a) + \dots + \frac{f^{(r-1)}(a)(x-a)^{r-1}}{(r-1)!} + \frac{f^{(r)}(\eta)}{r!} (x-a)^r, \quad a \leq \eta \leq x, \quad (8)$$

These are valid for $f(x) \in W[a, b]$ and $f(x) \in C[a, b]$, so Gelder's inequality also

$$\int_a^b |f(x)| \cdot |g(x)| dx \leq \|f(x)\|_{L[a,b]} \|g(x)\|_\infty. \quad (8')$$

3. Estimating the error of the local interpolation spline function in the domain $W^1[a_1, b_1]$

$$R_N(f; x) = \sum_{j=0}^3 \varphi_{j+1}(t) f(x_{i+j-1}) - f(x), \quad (9)$$

here

$$\begin{aligned} \varphi_1(t) &= \frac{11t^2}{12} - \frac{5t}{12} - \frac{t^3}{2} & \varphi_2(t) &= 1 - \frac{9t^2}{4} - \frac{t}{4} + \frac{3t^3}{2} \\ \varphi_3(t) &= \frac{7t^2}{4} + \frac{3t}{4} - \frac{3t^3}{2} & \varphi_4(t) &= \frac{t^3}{2} - \frac{5t^2}{12} - \frac{t}{12} \end{aligned}$$

Theorem 1. For the error $R_N(f; x)$ of the cubic spline $S_3(f; x)$ determined in (9), it is reasonable to estimate as follows [6,7]:

$$\max_{a \leq x \leq b} |S_3^{(r)}(x) - f^{(r)}(x)| \leq R_r, \quad r = 0, 1, 2,$$

In this $a_1 = a - h, \quad b_1 = b + h$ a R_r given in Table 1.

Table 1.

A class of functions	R_0	R_1	R_2
$W^1[a_1, b_1]$	$0,75hM_1$		

Proof. 1. We consider the proof of the theorem for the given classes.

For easy verification, the following equality holds:

$$\sum_{j=0}^3 \varphi_{j+1}(t) = 1, \quad (10)$$

In this case, let $f(x) \in W^1[a_1, b_1]$. Putting $r=l$, $a=x$, $x = x_{i+j-1}$ in the formula (7), we get the following:

$$f(x_{i+j-1}) = f(x) + \int_x^{x_{i+j-1}} f'(\vartheta) d\vartheta.$$

Applying this equality for $j=0,1,2,3$, we determine from (9).

$$R_N(f; x) = \sum_{j=0}^3 \varphi_{j+1}(t) [f(x) + \int_x^{x_{i+j-1}} f'(\vartheta) d\vartheta] - f(x).$$

and from that

$$R_N(f; x) = f(x) \sum_{j=0}^3 \varphi_{j+1}(t) + \sum_{j=0}^3 \varphi_{j+1}(t) \int_x^{x_{i+j-1}} f'(\vartheta) d\vartheta - f(x).$$

Applying equality (10) and using Gelder's inequality, we get:

$$\begin{aligned} |R_N(f; x)| &= \left| \sum_{j=0}^3 \varphi_{j+1}(t) \int_x^{x_{i+j-1}} f'(\vartheta) d\vartheta \right| \leq \sum_{j=0}^3 \left| \varphi_{j+1}(t) \int_x^{x_{i+j-1}} d\vartheta \right| \|f'(x)\|_\infty = \\ &= M_1 \sum_{j=0}^3 |\varphi_{j+1}(t)(x_{i+j-1} - x)|. \end{aligned}$$

$x_{i+j-1} - x = (j-1-t)h$, for being

$$|R_N(f; x)| \leq [|\varphi_1(t)|(1+t) + |\varphi_2(t)|t + |\varphi_3(t)|(1-t) + |\varphi_4(t)|(2-t)] M_1 \cdot h$$

If we consider the pointers of $\varphi_i(t)$, $i = \overline{1,4}$ s, we get the following

$$|R_N(f; x)| \leq [-\varphi_1(t)(1+t) + \varphi_2(t)t + \varphi_3(t)(1-t) - \varphi_4(t)(2-t)] h M_1$$

It follows that:

$$|R_N(f; x)| \leq t(1-t)g_2(t)M_1h \leq g_2 \cdot hM_1, \quad g_2(t) = 2(1+2t-2t^2),$$

$$g_2 = \max_{0 \leq t \leq 1} \{2t(1-t)(1+2t-2t^2)\}.$$

Then,

$$\|R_N(f; x)\| \leq 0,75M_1 \cdot h. \tag{11}$$

we will have [8].

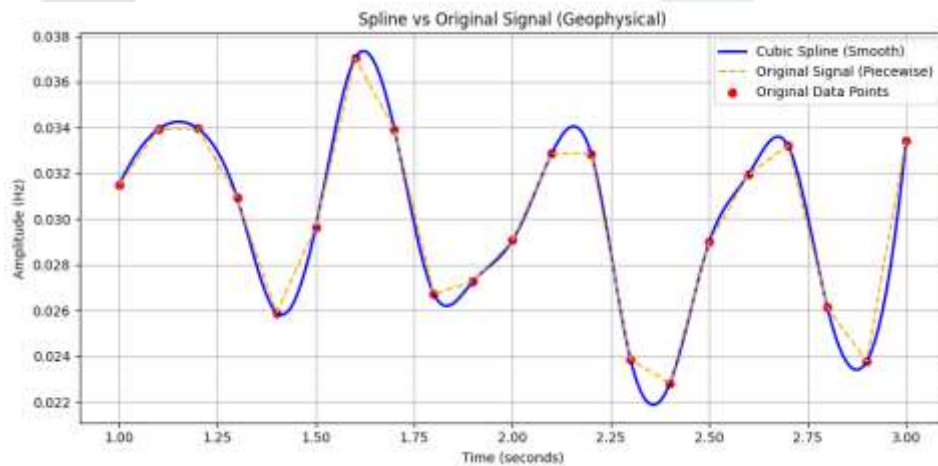
4. Application to digital signal processing based on the considered model.

The reconstruction of the geophysical signal, as shown in Table 2, was carried out using the developed cubic spline model. A third-order spline interpolation algorithm with a defect of two was implemented in MATLAB and applied for digital signal processing purposes (see Fig. 1). In practical settings, particularly in industrial applications, challenges such as signal degradation, interference contamination, and poor reception from specific zones are common during the transmission and acquisition of geophysical data. The subsequent analysis and experiments demonstrate that the proposed spline-based model performs effectively in addressing such issues, making it a valuable tool for enhancing the quality and reliability of digital signal processing in geophysical contexts.

Table 2. Amplitude values of geophysical signals over time.

No	x	Original y	Spline y	Error
1	1.1917	0.033995	0.034059	6.5e-05
2	1.2118	0.03188	0.033802	0.001922
3	1.2188	0.033129	0.033687	0.000557
4	1.2724	0.032198	0.032206	9e-06
5	1.2744	0.028587	0.032129	0.003541
6	1.3014	0.027369	0.030866	0.003497
7	1.3229	0.030213	0.029606	0.000607
8	1.3576	0.027837	0.027531	0.000305
9	1.4459	0.026699	0.026424	0.000275
10	1.4818	0.030314	0.028316	0.001998

Figure 1. Plot of geophysical signal recovery based on constructed cubic spline function.



As demonstrated by the results above, the digitally reconstructed geophysical signal using the proposed cubic spline model closely matches the original signal values. This indicates that, in future investigations involving object-specific geophysical challenges, scientific and practical research based directly on the spline model may provide a highly effective route toward achieving accurate and reliable results [10,11].

Conclusion:

In summary, spline theory continues to play a significant role in advancing science, particularly in the resolution of complex practical problems. This is clearly reflected in the present study. The formulation and implementation of modern, efficient models to address pressing challenges—such as meeting material demands, identifying and utilizing underground mineral resources—are regarded as key priorities in today’s scientific and industrial landscape. The findings of this work suggest that applying spline models offers a powerful tool in solving such issues. Specifically, the experimental results from geophysical signal recovery and digital processing show a maximum absolute error of 0.00354 Hz, highlighting the precision and reliability of the approach. Such promising outcomes further reinforce the importance of exploring spline-based methods, as each successful application contributes to scientific progress and the broader development of human knowledge and technology.

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