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ON IRREDUCIBLE HYPERFINITE REAL SUBFACTORS

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Abstract. *It is shown any hyperfinite factor has an involutive *-antiautomorphism. It is proved that a real subfactor is irreducible if and only if its enveloping factor is irreducible. Using constructed examples in the complex case, as well as using an involutive *-antiautomorphism of a W*-algebra, examples of irreducible hyperfinite real subfactors with index larger than 4 are constructed.*

Key words. *Hyperfinite factor, W*-algebra, irreducible hyperfinite real subfactors, complex Hilbert space, commutant, coupling constant, involutive *-antiautomorphism, principal graphs.*

1. INTRODUCTION

In 1930's von Neumann and Murray introduced the notion of coupling constant for finite factors. In 1983 V.Jones suggested a new approach to this notion, defined the notion of index for type II_1 factors, and proved a surprising theorem on values of the index for subfactors (see [1]). He also introduced a very important technique in the proof of this theorem: the towers of algebras. Since then this theory has become a focus of many fields in mathematics and physics. In [2], H.Kosaki extended the notion of the index to an arbitrary (normal faithful) expectation from a factor onto a subfactor. While Jones' definition of the index is based on the coupling constant, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A.Connes [3] as well as the theory of operator-value weights due to U.Haagerup [4]. In [2] and [5] it was shown that many fundamental properties of the Jones index in the type II_1 case can be extended to the general setting. At the present time, the theory of index thanks to works by V.Jones, P.Loi, R.Longo, H.Kosaki and other mathematicians is deeply developed and has many applications in the theory of operator algebras and physics.

In parallel with the theory of an index of complex subfactors the theory of an index of real subfactors has also been intensively developed. In paper [6] real analogues of Jones' theory of index is considered. In particular, the notions of the real coupling constant and the index for finite real factors was introduced and investigated.

In the present paper the study of the notion of index of real factors are continued. It is shown that any hyperfinite factor has an involutive *-antiautomorphism. It is proved that a real subfactor is irreducible if and only if its enveloping factor is irreducible. Using examples constructed in the complex case, as well as using an involutive *antiautomorphism of W*-algebra, series of examples of irreducible hyperfinite real subfactors with index larger than 4 will be constructed in the work.

2. PRELIMINARIES

Let H be a complex Hilbert space, $B(H)$ denote the algebra of all bounded linear operators on H. The *weak (operator) topology* on $B(H)$ is the locally convex topology, generated by the seminorm of the form: $\rho(a)=|(\xi,a\eta)|,\ \ \xi,\eta\in H,\ \ a\in B(H)$. *$W$*-algebra* is a weakly closed complex *-algebra of operators on a Hilbert space H containing the identity operator **1**. Recall that W*-algebras are also called *von Neumann algebras*. The *center* $Z(M)$ of a W*-algebra M is the set of elements of M, commuting with each element from M. Elements of $Z(M)$ are called

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central elements. A W^*-algebra M is called *factor*, if $Z(M)$ consists of the complex multiples of **1**, i.e $Z(M)=\{\lambda\mathbf{1}, \lambda\in\mathbb{C}\}$. We say that a W^*-algebra M is *injective* if there exists a projection P from $B(H)$ onto M such that $\|P\|=\mathbf{1}$ and $P(\mathbf{1})=\mathbf{1}$. This is equivalent to the condition that M is *hyperfinite*, i.e., there exists an increasing sequence $\{M_n\}$ of matrix subalgebras of the algebra M containing 1I and such that the union $\cup_n M_n$ is weakly dense in M. A linear mapping $\alpha: M\to M$ is called a **-automorphism* (respectively a *-antiautomorphism) if $\alpha(x^*)=\alpha(x)^*$ and $\alpha(xy)=\alpha(x)\alpha(y)$ (respectively $\alpha(xy)=\alpha(y)\alpha(x)$), for all $x,y\in M$. A mapping α is called involutive if $\alpha^2=id$. A *-automorphism α is called *inner* if there exists a unitary u in M, such that $\alpha(x)=Adu(x)=uxu^*$, for all $x\in M$.

A real *-subalgebra $R\subset B(H)$ with **1** is called a *real* W*-*algebra*, if it is weakly closed and $R\cap iR=\{0\}$. The smallest (complex) W*-algebra M containing R coincides with its complexication $R+iR$, i.e. $M=R+iR$. It is known that R generates a natural involutive (i.e. of order 2) *-antiautomorphism α_R of M, namely $\alpha_R(x+iy)=x^*+iy^*$, where $x+iy\in M$, $x,y\in R$. In this case $R=\{x\in M:\alpha_R(x)=x^*\}$. Conversely, given a W*-algebra M and any involutive *-antiautomorphism α on M, the set $(M,\alpha)\ =\ \{a\in M:\alpha(a)\ =a^*\}$ is a real W*-algebra (see [7], [8], [9]).

3. THE INDEX OF SUBFACTORS

Let $M(\subset B(H))$ be a finite factor and let τ be the unique faithful normal tracial state of M. If α is an involutive *-antiautomorphism of M, then it is clear that τ is automatically α-invariant. Denote by $L^2(M)$ the completion of M with respect to the norm $\|x\|_2=\tau(x^*x)^{\frac{1}{2}}$. Similarly by $L^2(M,\alpha)$ we denote the completion of the real factor (M,α). Then we have $L^2(M,\alpha)+iL^2(M,\alpha)=L^2(M)$ (see [6]).

If $M(\subset B(H))$ is a finite factor with the finite *commutant* $M':=\{x\in B(H): xy=yx, \forall y\in M\}$, the *coupling constant* $dim_M(H)$ of M is defined as $tr_M(E_\xi^{M'})/tr_{M'}(E_\xi^{M})$, where ξ is a non-zero vector in H. ξ is independent on ξ. Similarly defined the notation of coupling constant for real finite factors. One has the following relations between $dim_{(M,\alpha)}(H_r)$, $dim_{(M,\alpha)}(H)$ *and* $dim_M(H)$

$dim_M(H)=dim_{(M,\alpha)}(H_r)=dim_{(M,\alpha)}(H)$ (see [6]).

Now, consider a subfactor $N\subset M$ and let α be an involutive *-antiautomorphism of M with $\alpha(N)\subset N$. The index of N in M, denoted by $[M:N]$ is defined as $dim_N(L^2(M))$, i.e. this is the coupling constant of N when regarded in the standard representation of M. Similarly, for real factors $R=(M,\alpha)$ *and* $Q=(N,\alpha)$, the index of Q in R, denoted by $[R:Q]$ or by $[(M,\alpha):(N,\alpha)]$, is defined as $dim_Q(L^2(R))$. This is also the coupling constant of Q when regarded in the standard representation of *R*. Between real and complex indices there is the following relation (see [6]).

$$[(M,\alpha):(N,\alpha)] = [M,N] \quad i.e. \quad [R:Q] = [R+iR:Q+iQ]. \qquad (1)$$

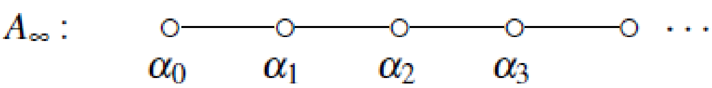
The first problem V.Jones considered in [1] was to characterize the set of all the possible values the index may take: he proved that [*M* : *N*] can only take the values:

$$\left\{4\cos^2\frac{\pi}{n} : n \geq 3\right\} \cup \{4;\ \infty\}$$

and that each of these values can appear.

4. MAIN RESULTS

The most interesting case of subfactors are those for which the relative commutant $N'\cap M$ is reduced to the scalars $\mathbb{C}\mathbf{1}$. V.Jones proved in [1] that small values of indices $[M:N]<4$ impose the condition $N'\cap M=\mathbb{C}$ (such subfactors are called *irreducible*). However, his examples of subfactors of index larger than 4 do not satisfy this condition (see. [1]), in fact they are of a certain trivial form. The problem of characterizing the values $[M:N]>4$ in the case $N'\cap M=\mathbb{C}$ remained open. First examples of hyperfinite factors $N\subset M$ with $N'\cap M=\mathbb{C}$ and index larger than 4 were given in [10]. Namely, it was constructed a series of irreducible hyperfinite subfactors of II_1 factor with lowest index: $3+\sqrt{3}=4.73205...$. U.Haagerup in [11] showed a list of candidates of (dual) principal graphs for the index range $(4,3+\sqrt{3}]$ by complicated and subtle combinatorial arguments. His work, in particular, implies that if the index value is between 4 and $(5+\sqrt{13})/2=4.302...,$ then the principal graph must be A_∞.



Haagerup-Schou and Ocneanu have constructed many subfactors of the hyperfinite II_1 factors with trivial relative commutants with indices in this range using bi-unitary connections on finite graphs. In this way, the Jones index we get is the square of the Perron Frobenius eigenvalue of the graphs we use. Then a result in the graph theory implies that the smallest index value above four we can construct in this way is 4.026..., even if we allow infinite graphs. Further, examples of irreducible hyperfinite subfactors with index larger than 4 were also obtained in the papers of H.Yoshida (see also [12], [13]).

Teorema 1. *Any hyperfinite factor has an involutive *-antiautomorphism.*

Proof. Let *N* be a hyperfinite factor and let *R* be a hyperfinite real factor. By [8] an algebra $M=R+iR$ is also hyperfinite factor and the map $\alpha: M\to M$ defined as $\alpha(a+ib)=a^*+ib^*$ $(a,b\in R)$ is involutive *-antiautomorphism of *M*, generating *R*, i.e. $R=(M,\alpha)=\{x\in M:\alpha(x)=x^*\}$. Since all hyperfinite factors of the same type are isomorphic between them, there exists an isomorphism $\theta: N\to M$. Then the map $\beta: N\to N$ defined as $\beta=\theta^{-1}\circ\alpha\circ\theta$ is involutive *-antiautomorphism of *N*. The theorem is proved.

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Theorem 2. There is series of irreducible hyperfinite real factors with index larger than 4, in particular, with index values in the intervals $(4, 3+\sqrt{3}]$ *and* $(4, \frac{5+\sqrt{13}}{2}]$.

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