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ON CONSTRUCTION OF REAL AW*-FACTORS

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ABSTRACT. The paper of the is to initiate the study of real AW*-algebras in the framework of the theory of real C*-algebras and W*-algebras. It happens that in some aspects real AW*-algebras behave unlike complex AW*-algebras and sometimes their properties are completely different also from corresponding properties of real W*-algebras. We prove that if the complexification $A+iA$ of a real C*-algebra A is a (complex) AW*-algebra then A itself is a real AW*-algebra. By modifying the Takenouchi's examples of complex non-W*, AW*-factors we show that there exist real non-W*, AW*-factors. The correspondence between real AW*-factors and involutive (i.e. with period 2) *-anti-automorphisms of (complex) AW*-factors is established. We give the decomposition of real AW*-algebras into types *I*, *II* and *III* similar to the case of complex AW*-algebras or W*-algebras. It is proved that if A is a real AW*-factor and its complexification $M=A+iA$ is also an AW*-algebra (and therefore an AW*-factor) thenthetypesof A and M coincide.

KEYWORDS: AW*-algebra, C*-algebra, factor, involutive *-antiautomorphism, complex Hilbert space, commutant, complexification, linear *-automorphism, conjugate, bicommutant, quaternions algebra, projection, isomorphic.

1. INTRODUCTION

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras *M* of the algebra *B*(*H*) of bounded linear operators on a complex Hilbert space *H*, which is closed in the weak operator topology. Equivalently *M* is a von Neumann algebra in *B*(*H*) if it is equal to the commutant of its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with trivial center and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types *I*, *II* and *III*.

Real operator algebra is a *-algebra consisting of bounded (real) linear operators on a real Hilbert space *H*. If it is closed in the weak operator topology we have real W*-algebra, and if it is uniformly closed (i.e. in the norm topology) then we come to the notion of the real C*-algebra.

In his monograph [7] Li Bing-Ren has set up the fundamentals of real operator algebras and gave a systematic discussion of the real counterpart for the theory of W*- and C*-algebras.

A slightly different (but almost the same up to *-isomorphism) definition of real W*-algebras was given by E.Størmer [13,14]: A real von Neumann algebra (or real W*-algebra) is a real *-algebra $\mathfrak{R}$ of bounded linear operators on a complex Hilbert space containing the identity operator **1**, which is closed in the weak operator topology and satisfies the condition $\mathfrak{R}\cap i\mathfrak{R}=\{0\}$.The smalles (complex) von Neumann algebra $U(\mathfrak{R})$ containing $\mathfrak{R}$ coincides with its complexification $\mathfrak{R}\cap i\mathfrak{R}$, i.e. $U(\mathfrak{R})=\mathfrak{R}\cap i\mathfrak{R}$. Moreover $\mathfrak{R}$ generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_{\mathfrak{R}}$ of $U(\mathfrak{R})$, namely $\alpha_{\mathfrak{R}}=(x+iy)=x^*+iy^*$, where $x+iy\in U(\mathfrak{R})$, $x,y\in\mathfrak{R}$. It is clear that $\mathfrak{R}=\{x\in U(\mathfrak{R}):\alpha(x)=x^*\}$. Conversely, given a (complex) von Neumann algebra *U* and any involutive *-antiautomorphism α on *U*, the set $\{x\in U:\alpha(x)=x^*\}$ is a real von Neumann algebra in the above sense.

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It is not diffucul to see that two real von Neumann algebras generating the same (complex) von Neumann algebra are isomorphic if and only if the corresponding involutive *-antiautomorphisms are conjugate. Thus the study of the above real von Neumann algebra can be reduced to the study of pairs (U, α), where U is a (complex) von Neumann algebra and α - its involutive *-antiautomorphism.

2. PRELIMINARIES

Let H be a complex Hilbert space, $B(H)$ denote the algebra of all bounded linear operators on H. The *weak (operator) topology* on $B(H)$ is the locally convex topology, generated by semi norms of the form: $\rho(a)=|(\xi,a\eta)|$, $\xi,\eta\in H$, $a\in B(H)$. W*-algebra is a weakly closed complex *-algebra of operators on a Hilbert space H containing the identity operator **1**. Recall that W∗-algebras are also called *von Neumann algebras*.

Let further M be a W*-algebra. The set M' of all elements from $B(H)$ commuting with each element from M is called the commutant of the algebra M. The *center* $Z(M)$ of a W*-algebra M is the set of elements of M, commuting with each element from M. It is easy to see that $Z(M)=M\cap M'$. Elements of $Z(M)$ are called *central* elements. A W*-algebra M is called *factor*, if $Z(M)$ consists of the complex multiples of **1**, i.e if $Z(M)=\{\lambda 1,\ \lambda\in\mathbb{C}\}$. We say that a W*-algebra M is *injective* if thereexists a projection P in $B(H)$ onto M such that $\|P\|=1$ and $P(1)=\mathbf{1}$. This isequivalent to the condition that M is *hyperfinite*, i.e., that there exists an increasing sequence $\{M_n\}$ of matrix subalgebras of the algebra M containing **1** and such that the union $\cup_n M_n$ is weakly dense in M.

Let e,f,h be projections from M. We say that e is equivalent to f, and write $e\sim f$, if $e=\omega^*\omega$, $f=\omega\omega^*$ for some partial isometry ω from M. A projection e is called: *finite*, if $e\sim f\le e$ implies $f=e$; *infinite* - otherwise; *purely infinite*, if e doesn't have any nonzero finite subprojection; abelian, if the algebra eMe is an abelian W∗-algebra. A W∗-algebra M is called *finite*, *infinite, purely infinite*, if **1** is a finite, infinite, purely infinite respectively; M is σ-finite, if any family of pairwise orthogonal projections from M is at most countable; *semifinite*, if each projection in M contains a nonzero finite subprojection; *properly infinite*, if every nonzero projection from $Z(M)$ is infinite; *discrete*, or of type *I*, if it contains a faithful abelian projection (i.e. an abelian projection with the central support **1**); *continuous*, if there is no abelian projection in M except zero; M is of type *II*, if M is semifinite and continuous; type I_{fin} (respectively I_∞), if M is of type *I* and finite (respectively properly infinite); type II_1 (respectively type II_∞), if M is of type *II* and finite (respectively properly infinite); type *III*, if M is purely infinite. It is known that any W*-algebra has a unique decomposition along its center into the direct sum of W*-algebras of the I_{fin}, I_∞, II_1, II_∞ and *III* types.

A linear mapping $\alpha:M\to M$ is called a **-automorphism* (respectively a *-antiautomorphism) if $\alpha(x^*)=\alpha(x)^*$ and $\alpha(xy)=\alpha(x)\alpha(y)$ (respectively $\alpha(xy)=\alpha(y)\alpha(x)$, for all $x,y\in M$. A mapping α is called involutive if $\alpha^2=id$. A *-automorphism α is called *inner* if there exists a unitary u in M, such that $Adu(x)=uxu^*$, for all $x\in M$. A *-automorphism is called centrally trivial if

$\alpha(x_n)-x_n\to 0$ *-strongly as $n\to\infty$ for any central sequence $\{x_n\}_{n\in\mathbb{N}}$. We shall denote by $Aut(M)$ the group of all *-automorphisms, by $Aut(M)$ the group of all *antiautomorphisms, by $Int(M)$ the group of all inner *-automorphisms, and by Ct(*R*) the subgroup of its centrally trivial *-automorphisms of *M*. Two *-automorphisms or *-antiautomorphisms α and β are said to be conjugate (or outer conjugate), if $\alpha=\theta\cdot\beta\cdot\theta^{-1}$ (respectively $Adu\cdot\alpha=\theta\cdot\beta\cdot\theta^{-1}$) for some *-automorphism θ (and an inner *-automorphism *Adu*). A linear functional ω on *M* is called *positive*, if $\omega(x^*x)\geq 0$ for all $x\in M$. A positive linear functional ω on *M* with $\|\omega\|=1$ is called a state. Let M_+ be the positive part of *M*. A *weight* on *M* is a homogeneous additive function $\omega: M_+\to[0,+\infty]$ (we suppose that $0\cdot+\infty=0$). A weight (or a state) ω is called: *faithful*, if for any $x\in M_+$, $\omega(x)=0$ implies $x=0$; *normal*, if for any net $\{x_\alpha\}$ in *M*, increasing to an element *x*, we have $\omega(x)=\sup_\alpha\omega(x_\alpha)$; *finite*, if $\omega(x)<\infty$ for all $x\in M_+$; *semifinite*, if for any $x\in M_+$ there exists a net of elements $\{y_\alpha\}\in M_+$, such that $\omega(y_\alpha)<\infty$, and $y_\alpha\to x$ in σ - weak topology; ω is a trace, if $\omega(uxu^*)=\omega(x)$ for all $x\in M_+$ and each unitary $u\in M$.

The type of a W*-algebra is tightly connect with the existence of traces on it. Namely a W*-algebra *M* is a finite if and only if it possesses a separating family of finite normal traces; it is semifinite if and only if it possesses a faithful semifinite normal trace; *M* is purely infinite if and only if there is no nonzero semifinite normal trace on M (see [15]).

Definition. [4]. By a real C*-algebra we mean a real Banach *-algebra *R* such that the relation $\|a^*a\|=\|a\|^2$ holds and the element $1+a^*a$ is invertible for any $a\in R$.

Definition*. [8,9]. A real C*-algebra *R* such that *R*+*iR* is a complex W*-algebra is referred to as a real W*-algebra.

We proceed with another definition of a real W*-algebra, which can be found in papers of Størmer.

Definition. [2,13].** A unital weakly closed real *-algebra *R* in $B(H)$ such that $R\cap iR=\{0\}$ is called a *real W*-algebra*.

A real W*-algebra *R* is called a (real) factor if its center *Z*(*R*) consists of elements $\lambda\mathbf{1}$, $\lambda\in\mathbb{R}$. We say that a real W*-algebra *R* is of type I_{fin}, I_∞, II_1, II_∞ and *III*, $\lambda\in[0,1]$ if the enveloping W*-algebra $U(R)=R+iR$ (i.e., the least W*-algebra containing *R*) is of the corresponding type with respect to the usual classification of W*-algebras.

3. MAIR RESULTS

Let A be a real C*-algebra, with the complexification $M=A+iA$. Then *M* is a complex C*-algebra and, as we have seen in the previous section, if A is a real AW*-algebra *M* may not be a (complex) AW*-algebra. Now let us consider the converse problem if $M=A+iA$ is an AW*-algebra is A necessarily a real AW*-algebra? The following result gives a positive answer to this problem.

Proposition 1. *Let A be a real C*-algebra and let* $M=A+iA$ *be its complexification. Suppose that M is an AW*-algebra. Then A is a real AW*-algebra.*

Proof. As we have mentioned in the first section, A coincides with the fixed point set under the conjugate linear *-automorphism $"-": x+iy \mapsto x-iy$ of *M*, where $x, y \in A$, i.e. $A=\{a \in M : \bar{a}=a\}$. If *S* is a nonempty subset in A then for its right-annihilator (with respect to *M*) we have

$$R_M(S)=\{a \in M \,|\, sa=0 \ \ for\ all\ s \in S\}$$

and

$$a \in R_M(S) \Leftrightarrow sa=0, \forall s \in S \Leftrightarrow \overline{sa}=\bar{s}\,\bar{a}=s\bar{a}=0, \ \forall s \in S$$

because $\bar{s}=s \in A$. This means that $a \in R_M(S)$ if and only if $\bar{a} \in R_M(S)$.

Now suppose that *M* is an AW*-algebra, then $R_M(S)=gM$ for a suitable projection $g \in M$. Since $g \in R_M(S)$ from above it follows that $\bar{g} \in R_M(S)$. Therefore $\bar{g}$ is a projection and $\bar{g} \in gM$, i.e. $\bar{g}=g\bar{g}$. Thus $(\bar{g})^*=(g\bar{g})^*=(\bar{g})^* g^*=\bar{g}g=\bar{g}=g\bar{g}$ i.e. $g=\bar{\bar{g}}=\overline{g\bar{g}}=\bar{g}g=\bar{g}$ This means that $g \in A$. But then

$$R_M(S)=R_M(S)\cap A=gM\cap A=gA,$$

i.e. A is a real AW*-algebra.

Proposition 2. *There exist real AW*-factors which are not real W*-factors.*

Theorem 1. *A real AW*-algebra A is a real W*-algebra if and only if*

(i) A possesses a separating family of normal states;

(ii) its complexification $M=A+iA$ is an AW-algebra.*

Proof. Necessity is obvious, since if A is a real W*-algebra, then $M=A+iA$ is a (complex) W*-algebra (see [7, Chap.5]). Therefore, *M* is an AW*-algebra and it possesses a separating family of normal states, the restrictions of which on *A* give a separating family of normal states on *A*.

Sufficiency. Let $M=A+iA$ be an AW*-algebra and let *A* possess a separating family of normal states, which we denote by $\{f_\gamma\}$, i.e. for any $a \in A$, $a \geq 0$, $a \neq 0$ exists $f \in \{f_\gamma\}$ with $f(a)=0$.

For $x \in a+ib \in M$, $a,b \in A$, we put $\alpha(x)=a^*+ib^*$. A straightforward calculation shows that α is an involutive (i.e. with period 2) *-anti-automorphism of *M*, and $A=\{a \in M : \alpha(a)=a^*\}$.

The extension of f_γ by linearity on *M* we denote by f_γ^0, and we shall show, that the family $\{f_\gamma^0\}$ is a separating family of normal states on *M*.

For $x=a+ib \in M_s=\{x \in M : x^*=x\}$ we have $a^*=a$, $b^*=-b$, and since f_γ is hermicitian we obtain $f_\gamma^0(x)=f_\gamma(a)+if_\gamma(b)=f_\gamma(a)$, since $f_\gamma(b)=0$. Thus, for $x \in M_s$ we have

$$f_\gamma^0(x)=\frac{1}{2}f_\gamma(x+\alpha(x)),$$

and $x+\alpha(x) \in A$, since $x+\alpha(x)=2a$.

If $x \geq 0$, then $\alpha(x) \geq 0$ (because *α* is a *-anti-automorphism), and hence $x+\alpha(x) \geq 0$, i.e. $x+\alpha(x) \in A^{+}$. Therefore $f_{\gamma}^{0}(x)=\frac{1}{2} f_{\gamma}(x+\alpha(x)) \geq 0$, i.e. all functionals f_{γ}^{0} are positive on *M*. Moreover, we have $f_{\gamma}^{0}(1)=f_{\gamma}(1)=1$, i.e. $\left\{f_{\gamma}^{0}\right\}$ is a family of states on *M*.

Now let us show, that each state f_{γ}^{0} is normal. If $\{x_{\nu}\} \subset M$ is an arbitrary net with $x_{\nu} \searrow 0$, then since *α* is an order isomorphism of *M*, we have $\alpha(x_{\nu}) \searrow 0$. Therefore $x_{\nu}+\alpha(x_{\nu}) \searrow 0$ and $x_{\nu}+\alpha(x_{\nu}) \in A^{+}$. Since f_{γ} is a normal we obtain

$$f_{\gamma}^{0}(x_{\nu})=\frac{1}{2} f_{\gamma}(x_{\nu}+\alpha(x_{\nu})) \to 0$$

i.e. all functionals f_{γ}^{0} are normal on *M*.

Finally, let $x \in M$, $x \geq 0$ and $f_{\gamma}^{0}(x)=0$ for all γ. Then $x+\alpha(x) \in A^{+}$, and since $\left\{f_{\gamma}\right\}$ is a separating family of states, $x+\alpha(x)=0$. Hence we have $x=-\alpha(x) \in M^{+} \cap(-M^{+})=\{0\}$, i.e. $x=0$. Thus, the AW*-algebra *M* possesses a separating family of normal states $\left\{f_{\gamma}^{0}\right\}$. By the theorem of Pedersen [10] *M* is a W∗-algebra. Therefore, by [7] *A* is a real W∗-algebra.

Now, let *M* be a (complex) AW*-factor, *α* its involutive *-anti-automorphism. Then as it was mentioned above the set $A=\{\alpha \in M: \alpha(a)=a^{*}\}$ is a real C*-algebra such that $M=A+iA$ (actually $\bar{x}=\alpha(x^{*})$ in terms of operation ”-”) and from Proposition 4.3.1 it follows that *A* is a real AW*-factor. It is known that two real W*-algebras generating the same (complex) W*-algebra, are isomorphic if and only if the corresponding involutive *-anti-automorphisms are conjugate [2,13,14]. A similar result is also valid for real AW*-algebras:

Proposition 3. *Let α and β be involutive *-anti-automorphisms of a (complex) AW*-factor M. Then the real AW*-factors*

$$A=\{x \in M: \alpha(x)=x^{*}\} \text{ and } B=\{x \in M: \beta(x)=x^{*}\}$$

*are real *-isomorphic if and only if the involutive *-anti-automorphisms α and β are conjugate, i.e.* $\beta=\theta \alpha \theta^{-1}$ *for a suitable *-automorphism of the AW*-factor M.*

Proof. Let *A* and *B* be real *-isomorphic with a *-isomorphism $\theta_{0}: A \mapsto B$. Then θ_{0} can be naturally extended to a (complex) *-isomorphism *θ* of their complexifications $A+iA$ and $B+iB$ both coincide with *M*. Therefore *θ* is a *-automorphism of *M* and $\theta(A)=B$, i.e. $\alpha(x)=x^{*}$ if and only if $\beta(\theta(x))=(\theta(x))^{*}=\theta x^{*}$. Thus for $x \in A$ we have

$$\beta(\theta(x))=(\theta(x))^{*}=\theta(x^{*})=\theta \alpha(x) \text{ , i.e. } \beta=\theta \alpha \theta^{-1} \alpha(x) \text{ for all } x \in A.$$

Since $\theta^{-1} \beta^{-1} \theta \alpha$ is a *-automorphism on *M* which is identical on *A* and any real *automorphism of *A* can be uniquely extended to a complex *-automorphism of *M*, it follows that $\theta^{-1} \beta^{-1} \theta \alpha=id$ on whole *M*, i.e. $\theta \alpha=\beta \theta$ and $\beta=\theta \alpha \theta^{-1}$, i.e. *α* and *β* are conjugate.

Conversely, if α and β are conjugate, i.e. $\beta = \theta\alpha\theta^{-1}$ for a suitable complex *-automorphism θ of *M*, then $\theta\alpha = \beta\theta$ and $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = \theta x^* = (\theta x)^*$, i.e. $\theta(A) = B$. Therefore, θ restricted on *A* gives the needed *-isomorphism between real AW∗-factors *A* and *B*.

Now we consider one of the main results of this section.

Theorem 2. *Let A be a real AW*-algebra and its complexification* $M = A + iA$ *is a (complex) AW*-algebra. Then A is of type I if and only if M of type I.*

Corollary. *Let A be a real AW*-algebra of type I, and its complexification* $M = A + iA$ *is a (complex) AW*-algebra. Then A is a real W*-algebra if and only if its center* Z_A *is a real W*-algebra.*

Proof. If *A* is a real W*-algebra then, obviously, its center Z_A is a real W*-algebra. Conversely, let *A* be an AW*-algebra of type *I* and its center is a W*-algebra. Then by Theorem 4.5.2, $M = A + iA$ is an AW*-algebra of type *I*, and its center $Z_M = Z_A + iZ_A$ is a W*-algebra. From Kaplanskys theorem [6, Theorem 2] it follows that *M* is an W*-algebra. Therefore, *A* is a real W*-algebra.

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